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Exact likelihood ratio test of the scale for censored Weibull sample

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Abstract

In the present paper the exact likelihood ratio test for the censored sample from Weibull distribution is derived and discussed. We discuss separately Type I, Type II and progressively censored samples. We are showing that the Type I and Type II censoring differ substantially. While in the case of Type II and progressively Type II censoring we are able to construct the pivotal quantity in the case of Type I censoring it is natural to use the exact likelihood ratio test, since no pivotal quantity is available here. The examples illustrate the methods developed here.

Keywords: exact test, likelihood ratio, Type I censoring, Type II censoring, progressive censoring, Lambert W function

1 Introduction

Censored sampling arises in a life-testing experiment whenever the experimenter does not observe the failure times of all units placed on a life-test. For case of progressive censoring see (Balakrishnan and Aggarwala (2000)).

The Weibull distribution has in recent years assumed a position of prominence in the field of reliability and life testing where samples are often either truncated or censored. Various topics associated with this distribution have been considered by numerous writers, among these are (Cohen (1965); Cohen (1975); Cohen (1991)). Let us consider a sample from the Weibull distribution with density of the form

$$f(y|\alpha,\sigma) = \frac{\alpha}{\sigma^\alpha} y^{\alpha-1} \exp(-(\frac{y}{\sigma})^\alpha), \quad y > 0,$$

where $\alpha > 0$ is shape parameter and $\sigma > 0$ is a scale parameter. Weibull distribution has many applications in reliability, engineering, physics or finance.

We consider the exact LR test of the scale hypothesis

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0. \tag{1}$$

The exact likelihood ratio tests for the scale hypothesis of the form (1) for various distributions have been derived recently. In (Stehlík (2003)) the case of gamma distributed observations is considered. (Stehlík (2006)) studied the case of Weibull observations, (Stehlík (2008)) considered case of generalized two-parameter gamma and in (Stehlík (2007)) the case when the time-to-failure information is missing is presented. However, until now only a little is made for the case of censored samples. Such a topic is of high importance for the theory and practice. This paper is devoted to this aim. Here we consider a typical life test, where N specimens are placed on test, and the elapsed time is recorded as each failure occurs. Finally, at some predetermined, fixed time T, or after some predetermined, fixed number of sample specimens have failed, the test is terminated. In both cases, the data collected consist of m fully measured observation $\{x_i\}_{i=1}^m$ plus the information that $N - m$ specimens survived beyond the time of censoring, T. For Type I censoring, T is fixed and m is the observed value of a random variable. For Type II censoring, m is fixed and T is the observed value of a random variable, that is, the mth-order statistics in a random sample of size N. For both Type I and Type II censoring, the likelihood function may be written as

$$L = K \prod_{i=1}^{m} \left[\frac{\alpha x_i^{\alpha-1}}{\sigma^\alpha} \right] \exp \left[- \sum_{i=1}^{m} (\frac{x_i}{\sigma})^\alpha \right] [1 - F(T)]^{N-m},$$

where K is the constant that does not depend on the parameters.

The paper is organized as follows. The results for Type-II censoring based on spacings are providing the basic formulation and as to how to go about doing the exact test. This constitutes the next session. The results for Type-I censoring are discussed in the 3rd section and are more complicated since there is no pivotal statistics based on spacings and the exact distribution becomes more complicated. Finally progressive Type-II censoring proceeds very much like Type-II censoring because of the results on spacings. To maintain the continuity of explanation technicalities and the concept of the optimality in Bahadur sense are put into the Appendix.

2 Exact LR test of scale for the Type II censored sample

The following theorem is providing the likelihood ratio statistics and the cdf of the Wilks statistics when the shape parameter α is known and sample is Type II censored.

Theorem 1 *Let the shape parameter α is known. Then the log-likelihood ratio statistics of the test of hypothesis (1) under Type II censoring has the form*

$$-\ln \Lambda = G_m \left(\sum_{i=1}^{m} (\frac{x_i}{\sigma_0})^\alpha + c(\frac{T}{\sigma_0})^\alpha \right) - G_m(m),$$

where $G_m(x) = x - m \ln x, x > 0$ is the function introduced in (Stehlík (2003)). The exact cumulative distribution function of the Wilks statistics $-2\ln\Lambda$ of the LR test of the hypothesis (1) has under H_0 the form

$$F(\tau) = F_m \left(-mW_{-1}(-e^{-1-\frac{\tau}{2m}}) \right) - F_m \left(-mW_0(-e^{-1-\frac{\tau}{2m}}) \right), \ \tau > 0$$

where F_m^Γ is the cumulative distribution function of the gamma distribution with shape parameter equals to m and scale parameter 1. *Here we denote by $W_k(x)$ the value of kth branch of the Lambert W function at point x (see Appendix).*

Proof. We have (see (Stehlík (2003)))

$$F(\tau) = P(2G_m(Y) - 2G_m(m) < \tau) = H \left(-mW_{-1}(-e^{-1-\frac{\tau}{2m}}) \right) - H \left(-mW_0(-e^{-1-\frac{\tau}{2m}}) \right), \ \tau > 0,$$

where H is the cumulative distribution function of the random variable

$$V = \sum_{i=1}^{m} (\frac{x_i}{\sigma_0})^\alpha + c(\frac{T}{\sigma_0})^\alpha.$$

By the power transformation (since the shape parameter is assumed to be known) everything can be transformed to the exponential case. The power statistic will be gamma in this case, since the order statistics raised to the power is order statistics $X_{m:N}$ from exponential sample and thus the random variable V can be written as a sum of spacings (see (Sukhatme (1937))), which are independent exponential, and therefore the sum statistic will have a gamma distribution. That is

$$V = \sum_{i=1}^{m} X_{i:N} + (N-m)X_{m:N} \sim \Gamma(m,1).$$

Thus $H = F_m^{\Gamma}$. $\square$

We conclude that the exact LR tests for the right Type II censored sample is identical with the exact LR scale test based on m uncensored observations (see also (Epstein and Sobel (1954))). The exact power function of the LR test of hypothesis (1) can be easily constructed on the base of Theorem 1. The problem is more complicated in the case of Type-I censoring. In Type I censoring we have no pivotal statistics, so it is natural to use the ELRT statistics. Here one can consider two cases.

a) Conditioning with the number of observed failures m. The interpretation and usage of ELRT is different, since you force to observe a fixed number of failures in the given interval in future.

b) The "unconditional" (it is always conditioned by "at least one failure occurs") likelihood inference, which is discussed in the next section of the paper.

The following example illustrates the case of Type II censoring.

Example 1 *Here we consider Type II censoring and compare the "naive"* T_m *and ELRT statistics. Our setup is* $\sigma_0 = 1, m = 10,$ *and for the sake of simplicity of quantiles we fix the size of the test to be* 0.0266. *The power functions (see Figure 1) are independent on the shape parameter* α.

$$p_e(\sigma) = 1 - F_{10}^{\Gamma}(-10\sigma W_{-1}(-e^{-1.25})) + F_{10}^{\Gamma}(-10\sigma W_0(-e^{-1.25}))$$

$$p_{T_10}(\sigma) = 1 - F_{10}^{\Gamma}(18.5/\sigma) + F_{10}^{\Gamma}(4.3/\sigma)$$

We can conclude that the reliability engineer should take into the consideration in which range of scale parameters σ is he interested before choosing an appropriate statistics: there are situations in which ELRT statistics has higher power (dependently on the σ) and vice versa (see Figure 1).

3 Exact LR test of scale for the Type I censored sample

The following theorem is providing the cdf of the Wilks statistics when the shape parameter α is known and sample is Type I censored.

Theorem 2 *The cdf of the Wilks LR statistics*

$$-2\ln\Lambda = 2G_m\left(\sum_{i=1}^{m}(\frac{x_{i:N}}{\sigma_0})^{\alpha} + (N-m)(\frac{T}{\sigma_0})^{\alpha}\right) - 2G_m(m)$$

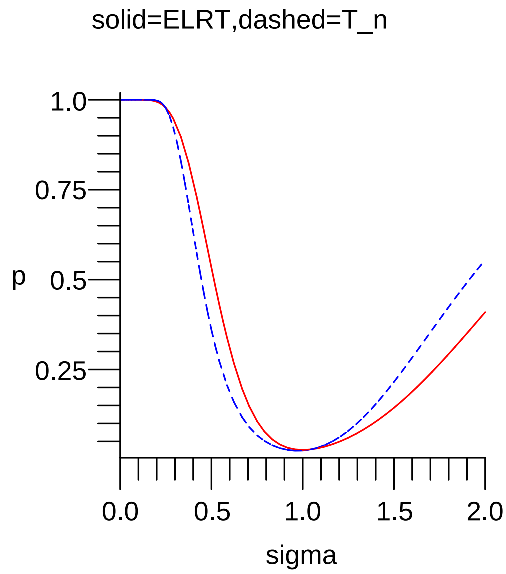

Figure 1: Comparison of powers

of the scale hypothesis (1) has the form

$$F(\tau) = \sum_{m=1}^{N} \left[H_m \left(-mW_{-1}(-e^{-1-\frac{\tau}{2m}}) \right) - H_m \left(-mW_0(-e^{-1-\frac{\tau}{2m}}) \right) \right] p(m), \;\; \tau > 0,$$

where H_m is the cdf of the random variable

$$Y_m = \sum_{i=1}^{m} (\frac{x_{i:N}}{\sigma_0})^{\alpha} + (N-m)(\frac{T}{\sigma_0})^{\alpha}.$$

Here $p(m)$ is truncated binomial distribution $b(N, 1-\exp(-\frac{T}{\sigma_0}))$ excluding 0, i.e.

$$p(m) = \frac{\begin{pmatrix} N \\ m \end{pmatrix} (1-\exp(-\frac{T}{\sigma_0}))^m \exp(-\frac{T(N-m)}{\sigma_0})}{1-\exp(-\frac{TN}{\sigma_0}).}$$

The pdf of Y_m can be adjusted from (Childs et al. (2003)) and has the form (here $0 < x < mNT)$:

$$h_m(x) = (1-q^N)^{-1} \sum_{k=0}^{m} \frac{C(k,m)}{m} g(\frac{x}{m} - T(k,m)^{\star}, \frac{m}{\sigma_0}, m), m < N$$

and

$$h_N(x) = (1-q^N)^{-1} \left[\frac{1}{m} g(\frac{x}{m}, \frac{N}{\sigma_0}, N) + \frac{N}{m} \sum_{k=1}^{N} \frac{(-1)^k q^k}{k} \begin{pmatrix} N-1 \\ k-1 \end{pmatrix} g(\frac{x}{m} - T(k,N)^{\star}, \frac{N}{\sigma_0}, N) \right].$$

where $C(k,d) = (-1)^k \begin{pmatrix} N \\ d \end{pmatrix} \begin{pmatrix} d \\ k \end{pmatrix} q^{N-d+k}$, $T(k,d)^\star = (N-d+k)T/d,\ q = e^{-T/\sigma_0}$ *and*

$$g(y,\gamma,v) = \begin{cases} \gamma^v \dfrac{y^{v-1}}{\Gamma(v)} \exp(-\gamma y), & \textit{for } y > 0, \\ 0, & \textit{for } y \leq 0. \end{cases}$$

is the pdf of gamma distribution. CDF has the form $(0 < x < mNT)$:

$$H_m(x) = 1 - (1-q^N)^{-1} \sum_{k=0}^{m} \frac{C(k,m)}{(m-1)!} \Gamma(m, A_m(T(k,m)^\star)),\ m < N$$

and

$$H_N(x) = 1-(1-q^N)^{-1} \left[\frac{\Gamma(N, \frac{x}{\sigma_0})}{(N-1)!} + \frac{N}{(N-1)!} \sum_{k=1}^{N} \frac{(-1)^k q^k}{k} \begin{pmatrix} N-1 \\ k-1 \end{pmatrix} \Gamma(N, A_N(T(k,N)^\star)) \right],$$

where $A_k(a) = k(x/m-a)^+/\sigma_0$ *and* $\Gamma(a,z) = \int_z^\infty t^{a-1} e^{-t} dt$ *is the incomplete gamma.*

Proof Let us fix the number of observed failures m, $m = 1, \ldots, N$. We obtain that for the cdf of the Wilks statistics we have

$$P(2G_m(Y_m) - 2G_m(m) < \tau) = H_m\left(-mW_{-1}(-e^{-1-\frac{\tau}{2m}})\right) - H_m\left(-mW_0(-e^{-1-\frac{\tau}{2m}})\right),\ \tau > 0,$$

where H_m is the cumulative distribution function of the random variable Y_m, which can be adjusted from (Childs et al. (2003)) by finding the distribution of $m\hat{\sigma}$ under the condition that m units are completely observed until failure. Notice, that despite the Type II censoring case Y_m has a finite support since $Y_m \leq N(\frac{T}{\sigma_0})^\alpha$. Also notice that h_m is not a pure mixture but a so called generalized mixture, since summand signs can alternate. Here $p(m)$ is truncated binomial distribution $b(N, 1-\exp(-\frac{T}{\sigma_0}))$ excluding 0. This completes the proof. □

The following Example 2 illustrates the usage of ELRT in a Type-I censoring scheme.

Example 2 Here we consider the exponential data given by (Bartholomew (1963)) and later elaborated by (Childs et al. (2003)) in the Type-II hybrid censoring scheme. The data are consisting of $N = 20$ items being put on a life test for a prefixed time of 150 hours resulting in the following observed failure times: $3, 19, 23, 26, 27, 37, 38, 41, 45, 58, 84, 90, 99, 109, 138$. In order to illustrate ELRT, we suppose that a censoring time of $T = 50$ was used, and we use for the null hypothesis the value of $\sigma_0 = 101.8$, given by (Childs et al. (2003)).

Thus we consider the testing problem

$$H_0 : \sigma = 101.8 \text{ versus } H_1 : \sigma \neq 101.8.$$

The Wilks statistics has the value $-2\ln\Lambda = 2G_9(7.9469) - 2G_9(9) = 33.29827857$. The exact power function of the hypothesis (1) for a observed data has the form

$$p(\sigma) = 1 - \sum_{m=1}^{20} \left[H_m\left(-mW_{-1}(-e^{-1-\frac{16.649}{m}})\right) - H_m\left(-mW_0(-e^{-1-\frac{16.649}{m}})\right)\right] p(m),$$

where H_m and p_m are computed at the given value of the alternative σ.

4 Exact LR test of scale for the progressively Type II censored sample

Progressive Type-II censoring is the versatile scheme of censoring studied thorougly by Balakrishnan and Aggarwala (2000). From a total of N units placed on a life-test only m are completely observed until failure. At the time of the first failure, R_1 of the $N-1$ surviving units are randomly withdrawn (or censored) from the life-testing experiment. At the time of the next failure, R_2 of the $n-2-R_1$ surviving units are randomly withdrawn (or censored), and so on. Finally, at the time of the m-th failure, all the remaining $R_m = N - m - R_1 - \ldots - R_{m-1}$ surviving units are censored. Censoring takes place here progressively in m stages. This scheme includes as special cases the complete sample situation (when $m = N$ and $R_1 = R_1 = \ldots = R_m = 0$) and the conventional Type II right censoring situation (when $R_1 = \ldots = R_{m-1} = 0$ and $R_m = N - m$). The inference for the one parameter exponential (what corresponds to the one parameter Weibull with the known shape parameter) for the progressively Type II Right $(R_1, ..., R_m)$ censored sample is equivalent with the inference based on the complete sample with size m (see [Viveros and Balakrishnan (1994)] for some estimation aspects). Regarding the ELRT it can be easily seen from the following consideration.

The Wilks log-likelihood ratio statistics of the test of hypothesis (1) under progressive right Type II censoring has the form

$$-2\ln\Lambda = 2G_m\left(\sum_{i=1}^{m}(R_i+1)(\frac{x_{i:m:N}}{\sigma_0})^{\alpha}\right) - 2G_m(m).$$

We have

$$F(\tau) = P(2G_m(Y) - 2G_m(m)) < \tau) = H\left(-mW_{-1}(-e^{-1-\frac{\tau}{2m}})\right) - H\left(-mW_0(-e^{-1-\frac{\tau}{2m}})\right),\ \tau > 0,$$

where H is the cumulative distribution function of the random variable

$$Y = \sum_{i=1}^{m}(R_i+1)(\frac{x_{i:m:N}}{\sigma_0})^{\alpha}.$$

By the power transformation (since the shape parameter is assumed to be known) everything can be transformed to the exponential case. $H = F_m^{\Gamma}$ since the random variable Y can be written as a sum of progressively Type II right censored spacings which are independent exponential (see Thomas and Wilson, 1972). For a special case of no censoring ($R_1 = \ldots = R_m = 0$) we obtained the spacings introduced by Sukhatme (1937).

We conclude that the exact LR tests for the progressively right Type II censored sample is identical with the exact LR scale test based on m uncensored observations (see also (Epstein and Sobel (1954))). This particularly means that scale test is asymptotically optimal in the Bahadure sense (AOBS) (see Appendix). For the case of general progressively Type II censored samples, even for the one parameter exponential distribution (with $r > 0$), the MLE of the scale parameter does not exist in an explicit form and has to be determined by a numerical method (see (Balakrishnan and Sandhu (1996))). For the case $r = 0$ there is an explicit solution.

However, the general progressive censoring when $r > 0$ is not of practical interest and only the case $r = 0$ is of interest (see (Balakrishnan (2007))). The progressively Type II censoring is much simpler case than the progressively Type I censoring, since we can construct the pivotal statistics on the base of spacings. However, in the progressively Type I censoring such a construction is not available. Thus the natural solution is to construct the Exact likelihood ratio test. This will be our interest in future.

5 Appendix

5.1 Lambert W function

The Lambert W function is defined to be the multivalued inverse of the complex function $f(y) = ye^y$. As the equation $ye^y = z$ has an infinite number of solutions for each (non-zero) value of $z \in \mathbf{C}$, the Lambert W has an infinite number of branches. A detailed discussion of the branches of the Lambert W can be found in (Corless et al. (1996)). The branches of the LW function are implemented to many mathematical computational softwares, e.g. the Maple, Matlab, Mathematica.

5.2 AOBS

Consider a testing problem $H_0 : \vartheta \in \Theta_0 \ vs \ H_1 : \vartheta \in \Theta_1 \setminus \Theta_0$, where $\Theta_0 \subset \Theta_1 \subset \Theta$. Further consider sequence $T = \{T_N\}$ of test statistics based on $y_1, ..., y_N \ iid \sim P_\vartheta, \vartheta \in \Theta$ We reject for large values of test statistics.

For ϑ and t denote $F_N(t, \vartheta) := P_\vartheta\{s : T_N(s) < t\}; \quad G_N(t) := \inf\{F_N(t, \vartheta) : \vartheta \in \Theta_0\}$. The quantity $L_n(s) = 1 - G_n(T_n(s))$ is called the attained level or the p-value. Suppose that for every $\vartheta \in \Theta_1$ the equality

$$\lim \frac{-2 \ln L_n}{n} = c_T(\vartheta)$$

holds a.e. P_ϑ. Then the nonrandom function c_T defined on Θ_1 is called the Bahadur exact slope of the sequence $T = \{T_n\}$.

Raghavachari (1970) and Bahadur (1971) have proved

$$c_T(\vartheta) \leq 2I(\vartheta, \Theta_0) \tag{2}$$

holds for each $\vartheta \in \Theta_1$. Here $I(\vartheta, \Theta_0) := \inf\{I(\vartheta, \vartheta_0) : \vartheta_0 \in \Theta_0\}$, where $I(\vartheta, \vartheta_0)$ is the Kullback Leibler divergence between ϑ and ϑ_0.

If (2) holds with the equality sign for all $\vartheta \in \Theta_1$, then the sequence T is said to be asymptotically optimal in the Bahadur sense. The maximization of $c_T(\vartheta)$ is a nice statistical property, because the greater the exact slope is, the more one can be convinced that the rejected null hypothesis is indeed false. The class of such statistics is apparently narrow, though it contains under certain conditions the LR statistics (see Bahadur, 1965, Bahadur (1967), Rublík, 1989a and b). Rublík proved asymptotical optimality of the LR statistic under regularity condition which is shown to be fullfiled by regular normal, exponential and Laplace distribution under additional assumption that Θ_0 is a closed set and Θ_1 is either closed or open in metric space Θ. In Stehlík (2003) is proved, that the homogeneity and scale test is AOBS in the case of observations distributed exponentially.

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